Bachelor of Science (B.Sc.) Semester—VI (C.B.S.) Examination

M_{11} : ABSTRACT ALGEBRA

Paper—1

(Mathematics)

Time : Three Hours] [Maximum Marks : 60

N.B. :— (1) Solve all the **FIVE** questions.

(2) All questions carry equal marks.

(3) Question Nos. **1** to **4** have an alternative. Solve each question in full or its alternative in full.

UNIT—I

1. (A) If G is a group and a(G) is the set of automorphisms of G, then prove that a(G) is a group under the composition of functions in a(G). 6

(B) Let G be a group; for $g \in G$ define a mapping $T_g : G \to G$ by $T_g(x) = g^{-1} x g$ for all $x \in G$. Prove that T_g is an automorphism of G. 6

OR

(C) Prove that conjugacy is an equivalence relation on a group G. 6

(D) Let Z be the centre of a group G. Then prove :

(i) $a \in Z \Leftrightarrow N(a) = G$ and

(ii) If Z is finite, then

$a \in Z \Leftrightarrow o(N(a)) = o(G)$.

where N(a) is normalizer of a in group G. 6

UNIT—II

2. (A) Prove that a nonempty subset S of a vector space V over the field F is a subspace of V if and only if the following conditions are satisfied :

(a) If $u, v \in S$, then $u + v \in S$

(b) If $u \in S$ and α a scalar, then $\alpha u \in S$. 6

(B) Find the span of a subset S of vector space V_3, where $S = \{(1, 2, 1), (1, 1, -1), (4, 5, -2)\}$, and prove that $(2, -1, 8) \in [S]$ but $(1, -3, 5) \notin [S]$. 6

OR

(C) Prove that in an n-dimensional vector space V, any set of n linearly independent vectors is a basis. 6

(D) Let {(1, 0, 1, 0), (0, –1, 1, 0)} be a linearly independent subset of the vector space V_4. Extend it to a basis for V_4. 6

UNIT—III

3. (A) Let $T : V_2 \rightarrow V_4$ be a Linear map defined by T(1, 1) = (0, 1, 0, 0), T(1, –1) = (1, 0, 0, 0) where {(1, 1), (1, –1)} is a basis of V_2. Then find T(x, y). 6

(B) Let $T : U \rightarrow V$ be a Linear map. Then prove that T is one-one if and only if $N(T) = \{\bar{o}_4\}$. Hence show that a Linear map $T : V_3 \rightarrow V_4$ defined by $T(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, x_3)$ is one-one. 6

OR

(C) Let $T : V_4 \rightarrow V_3$ be a Linear map defined by $T(e_1) = (1, 1, 1)$, $T(e_2) = (1, -1, 1)$, $T(e_3) = (1, 0, 0)$, $T(e_4) = (1, 0, 1)$, where $\{e_1, e_2, e_3, e_4\}$ is a standard basis of V_4. Verify Rank-Nullity theorem. 6

(D) Prove that a Linear map $T : U \rightarrow V$ is nonsingular if and only if there exists a Linear map $S : V \rightarrow U$ such that $TS = I_V$ and $ST = I_U$, where I_V is the identity function on V and I_U is the identity function on U. 6

UNIT—IV

4. (A) Find the matrix of the Linear transformation $T : V_2 \rightarrow V_3$ defined by :

$$T(x_1, x_2) = (-x_1 + 2x_2, x_2, -3x_1 + 3x_2)$$

related to the bases $B_1 = \{(1, 2), (-2, 1)\}$ and $B_2 = \{(-1, 0, 2), (1, 2, 3), (1, -1, 1)\}$. 6

(B) Find the range, kernel, rank and nullity of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix}$. 6

OR

(C) If V is an inner product space over F and u, v ∈ V, then prove Cauchy-Schwarz inequality $|u \cdot v| \le \|u\| \, \|v\|$. 6

(D) Using Gram-Schmidt orthogonalization process, orthonormalize the linearly independent subset {(1, 1, 1), (0, 1, 1), (0, 0, 1)} of V_3. 6

Question—V

5. (A) Show that $I(G) = \{I\}$ for an abelian group G, where I(G) is the set of inner automorphisms of G. 1½

(B) State Cayley's theorem for group. 1½

(C) Prove that in a vector space V over F $(-1)\,u = -u, \forall\ u \in V$. 1½

(D) If U and W are finite dimensional subspaces of a vector space V and $U + W = U \oplus W$, then show that $\dim (U \oplus W) = \dim U + \dim W$. 1½

(E) Let U and V be vector spaces over a field F and $T : U \to V$ be a linear map. Then prove $T(\bar{o}_u) = \bar{o}_v$. 1½

(F) Let U, V be finite dimensional vector spaces and $T : U \to V$ be a linear, one-one and onto map. Then prove dim V = dim U. 1½

(G) In an inner product space V, prove that

$(u + v) \cdot w = u \cdot w + v \cdot w$, where $u, v, w \in V$. 1½

(H) If H is orthogonal matrix, then prove that $\det H = \pm 1$. 1½